GENERAL RELATIVITY

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Relativity on the Streets

GPS

∀ Accuracy of ~ 20 – 30 nanosec ~ few metres

∀ Satellite velocity ~ 14,000 km/hr, R ~ 20,000 km

- SR time-dilation 7 microsec per day
- GR 45 microsec per day
- 38 microsec gives 10 km off

∀ Relativitistic corrections a MUST for GPS to function accurately

Galilean Principle of Relativity

Laws of Mechanics are the same in all Inertial Frames (IF)

IF: If the net external force acting on a body is zero, it is possible to find a set of reference frames (moving uniformly with respect to each other) in which the body has no acceleration. These frames are called IF.

Newton's I law

Laws of physics are form invariant

Newton's II law:

$$\vec{F} = m\,\vec{a}$$

alilean transformations:

$$x' = x - vt, \quad y' = y, \quad z' = z$$

(Nobody talks of time – its obvious !!!)

- Force, mass, acceleration are invariant whatever IF we adopt
- Kinetic energy, velocities may be different but the LAWS

  remain the same - covariance

Range of Newtonian mechanics: Macro-molecules – Galaxies

Then came Electromagnetism …

Maxwells Equations

Their form does not remain the same under

Galilean transformations but under

Lorentz Transformations

Lorentz Transformations:

$$x' = (x - vt)\gamma, \quad y' = y, \quad z' = z, \quad ct' = (ct - v\,x/c)\gamma,$$

$$\gamma = (1 - v^2/c^2)^{-1/2}$$

What happens to Mechanics?

Einstein’s Solution:

Change mechanics so that its laws are
orm invariant under Lorentz transformations

New definition of momentum

Law of momentum conservation should hold in all IF

Newtonian defn: p = mass times velocity does NOT work

$$p_x = m_0 \frac{dx}{d\tau}$$

τ is the proper time as measured by the particle

$$p = m_0 \, v \, \gamma$$

With this defn momentum is conserved

Postulates of Special Relativity

- Principle of Relativity:

The laws of physics are the same for observers in all Inertial Frames (IF).

There is no experiment in physics that singles out a preferred intertial frame.

- Constancy of speed of light:

$$c \sim 3 \times 10^5 \text{ km/sec}$$

The speed of light c is the same in all Inertial Frames

STEP II: Special Relativity + Gravity

So far mechanics, electrodynamics (optics) has been Relativised - Gravity ?

The Newtonian theory: Inverse square law

$$\vec{F} = -G \frac{m_1 m_2}{r^2} \hat{r}$$

Gravity signals travel instantaneously

Inconsistent with SR (Never mind Mercury)

Need a new theory of gravity → General Theory of Relativity

General Relativity

Observation 1:

Newton's I law: A body moves in straight line ....

Einstein asks: Where are the straight lines ???

Earth, moon, satellites, stones, cricket balls – ALL move on curves !

Newton's I law holds but only approximately in small regions and for short time

So we cannot have global IFs but local IFs - LIFs

Gravity: Result of choosing a wrong frame?

Observation 2:

The force of gravity is mass proportional

But we know other forces having the same

property - centrifugal, coriolis

$$F_{cor} = m\, v \times \omega$$

$$F_{cent} = m\, r\, \omega^2$$

They appear in rotating frames but disappear otherwise

Q: Is the same true with gravity?

Answer: Yes! But with a caveat

Go to the freely falling frame ...

Gravity does disappear but in a small region

of SPACETIME

⟶ Local Inertial Frame (LIF)

Principle of Equivalence !

Principle of Equivalence

Principle of Equivalence (WEP):

All masses fall with the same acceleration

Experimental verification!

Principle of Equivalence (SEP):

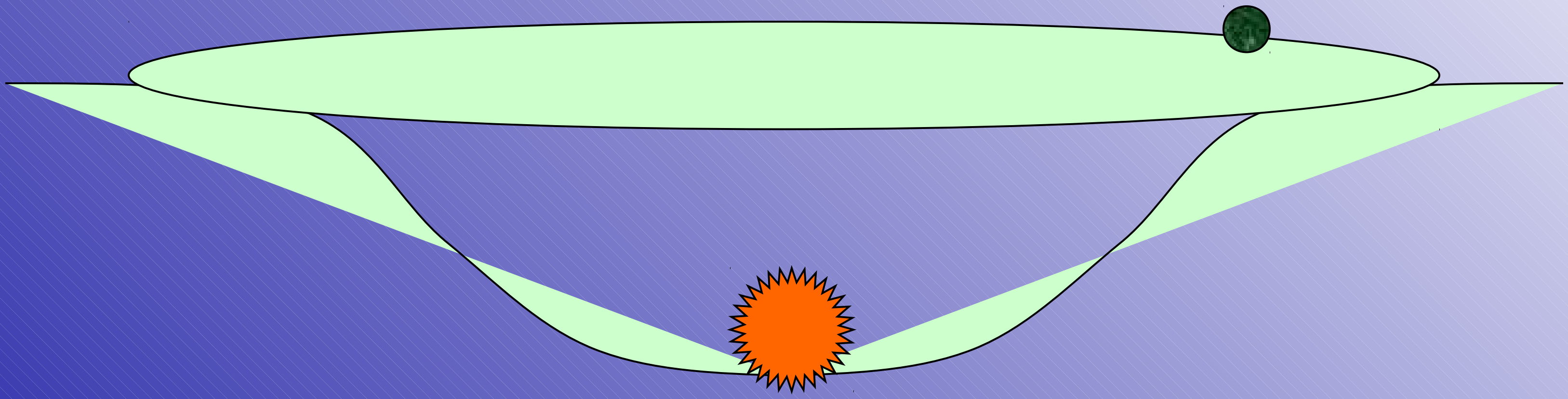
The laws of physics are the same in all LIFs

Curved Spacetime

Glue all the LIFs together - curved spacetime

CURVATURE ⟺ GRAVITY

EINSTEIN'S GRAVITY : General theory of Relativity



Gravitation:

Manifestation of the curvature of spacetime

Dynamical variables in General Relativity

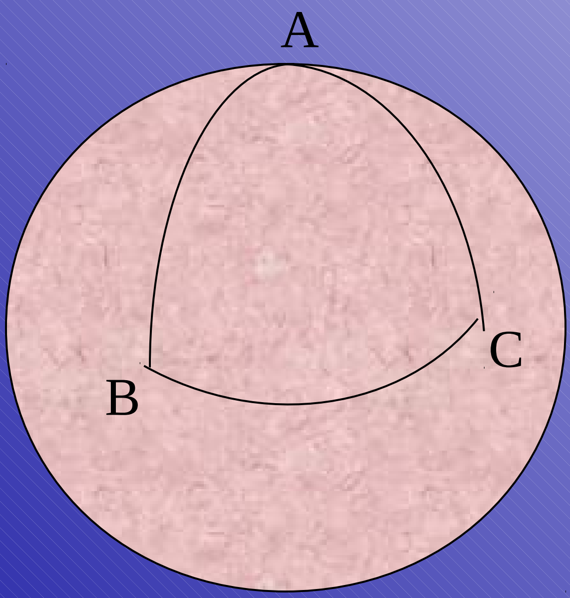
Dynamical variables:

Metric: g_{ij} → Newtonian potential ϕ

Curvature: R_{hijk} → Gravitational field

$\partial_i \partial_j \phi$: Tidal force

Curvature in 2 D



The 3 angles of the triangle do not add up to 180^o

Signature of curvature !

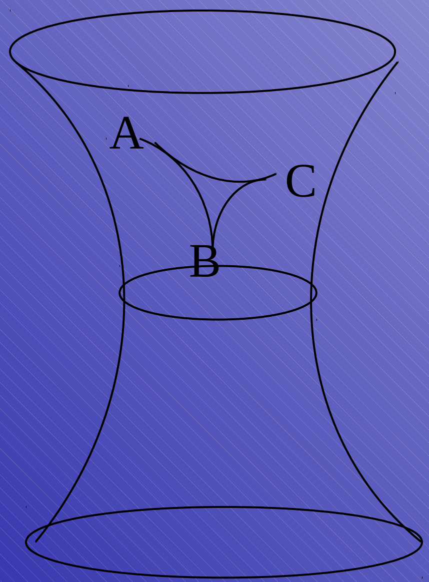
$$A + B + C = \pi + \delta$$

$$\delta = \int_{\Delta} K \, dS$$

K > 0

$$K = \frac{1}{R^2}$$

Negative curvature



$$K < 0$$

$$A + B + C < \pi$$

$$A + B + C = \pi + \delta$$

$$\delta < 0$$

4 – dimensional generalisations of metric and curvature

$$ds^2 = E\,du^2 + 2\,F\,dudv + G\,dv^2$$

$$\to ds^2 = g_{ij}\,dx^i\,dx^j$$

$$K \quad \to \quad R^i{}_{jkl} \qquad i, j, k, l = 0,1,2,3$$

Einstein's Field Equations

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Geometry = Matter distribution

Spacetime grips mass, telling it how to move, and mass grips spacetime, telling it how to curve

— John Archibald Wheeler

Compare with Newton's equation $\nabla^2 \phi = 4\pi G \rho$

Minkowski Spacetime

$T_{ik} \square 0$ throughout spacetime

$$ds^2 = c^2\, dt^2 - (dx^2 + dy^2 + dz^2)$$

$$R^i{}_{jkl} \equiv 0 \quad \rightarrow \quad \phi \equiv 0$$

The Newtonian limit

Weak field, slow motion limit:

$$T_{00} = \rho c^2 \quad >> \quad other \quad T_{ik}$$

Einstein's equations reduce to Newton's equation

$$\nabla^2 \phi = 4\pi G \rho$$

$$ds^2 = (1 + \frac{2\phi}{c^2}) c^2 dt^2 - (1 - \frac{2\phi}{c^2})(dx^2 + dy^2 + dz^2)$$

Schwarzschild Solution (1916)

Solution of Einstein's Equations for a non-rotating uncharged point mass

$$ds^2 = \left(1 - \frac{2m}{r}\right)c^2\, dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2\, d\,\Omega^2$$

$$m = \frac{M\,G}{c^2}$$

BlACKHOLE SOLUTION !

Cosmological Solutions: Robertson-Walker universes

$$ds^2 = c^2 dt^2 - S^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

$$k = \pm 1, 0$$

$$\dot{S}^2 + k = \frac{8\pi G}{3}\rho S^2$$ Einstein's equation

$$\frac{d}{dS}(\rho S^3) = -3 p S^2$$ Energy conservation equation

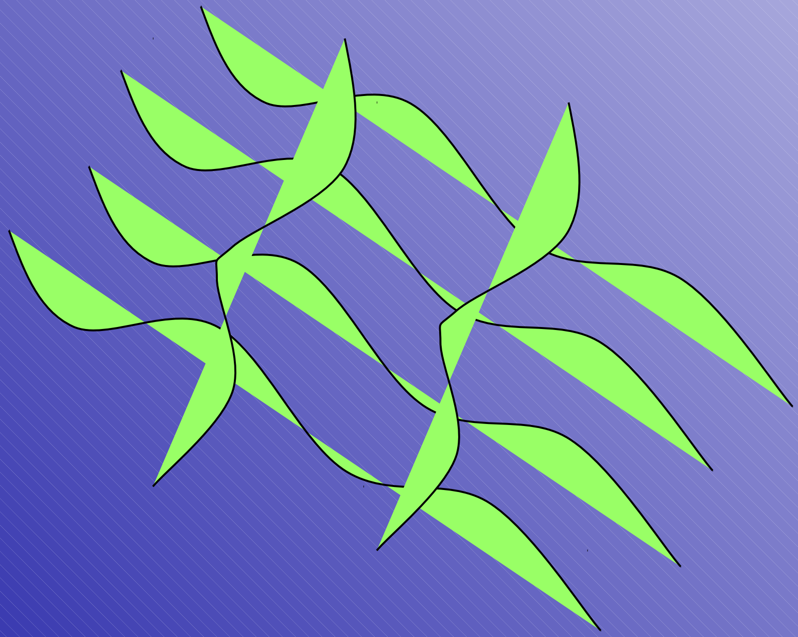
$$p = p(\rho)$$ Equation of state

p = 0 or dust: Friedmann solutions

MASSES OSCILLATE :

GRAVITATIONAL WAVES ???

Waves in the curvature of spacetime



Waves in curvature: K keeps flipping sign

In higher dimensions $R^{(0)}{}_{hijk}\ e^{-i\omega t}$

Wave solutions

Weak field : $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

Lorentz gauge : $$\bar{h}^{\mu\nu}{}_{,\nu} = 0$$

Linearised Einstein's Eq. : $$\Box \bar{h}_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu}$$

Trace reverse: $$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

Trace : $$h = h^{\mu}{}_{\mu}$$

Plane wave solutions: TT gauge

Source free: $\Box \bar{h}_{\mu\nu} = 0$

Plane wave solutions: $\bar{h}_{\mu\nu} = A_{\mu\nu}\, e^{i k_\alpha x^\alpha}$

Gauge conditions:

Transverse: $\bar{h}^{\mu\nu}{}_{,\nu} = 0 \quad \Rightarrow \quad A_{\mu\nu} k^\nu = 0$

Traceless: $h = h^\mu{}_\mu = 0 \quad \Rightarrow \quad A^\mu{}_\mu = 0$

$\exists \quad U^\mu \quad \Rightarrow \quad A_{\mu\nu} U^\nu = 0$

Transverse Traceless (TT) gauge: **Two polarisations**

GRAVITATIONAL WAVES

- Space-time warpage in the fabric of spacetime

  travels with the speed of light

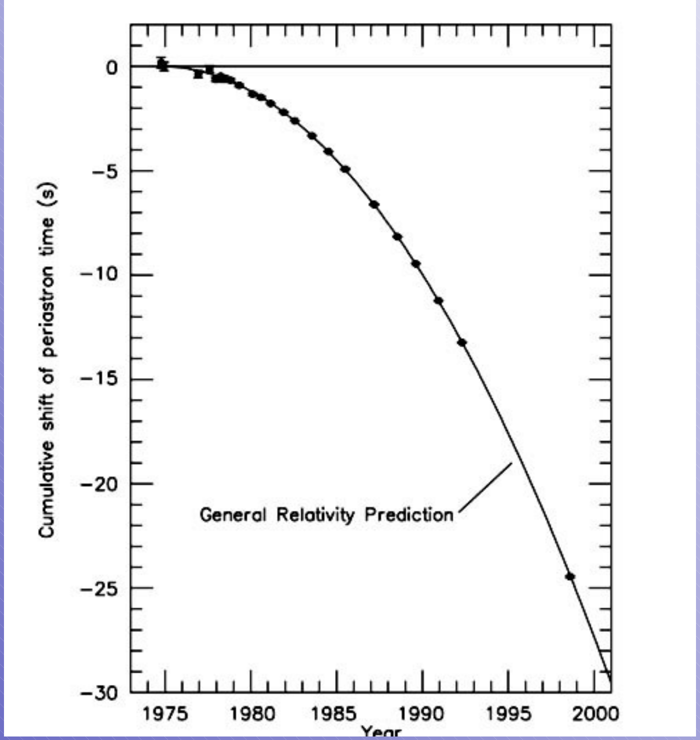
- Dynamic concentrations of matter

  Decay in the orbit of the

  binary pulsar PSR 1913+16

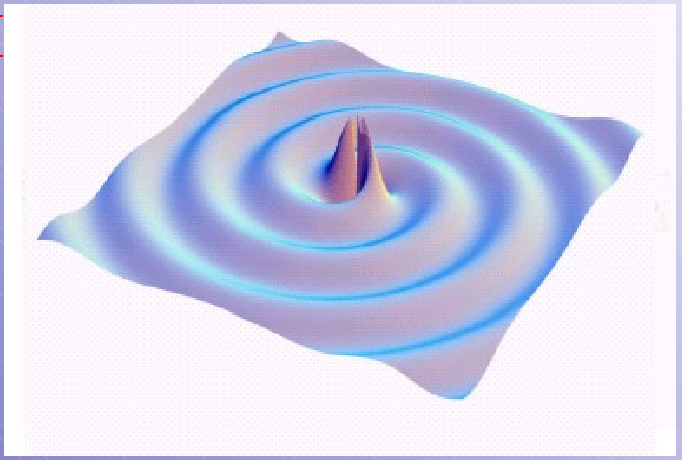
  - Nobel Prize to

  Hulse & Taylor 1993



Gravitational Waves EXIST !

GRAVITATIONAL WAVE ASTRONOMY



- Enormous differences between GW and EM

- Produced by bulk motions of matter

- Compact objects: Blackholes, neutron stars

- Not easily scattered: Hi fidelity info

- EM (f > 10^7 Hz) while GW (f < 10^4 Hz)

-Information orthogonal to EM - revolution

PROBES OF THE UNIVERSE

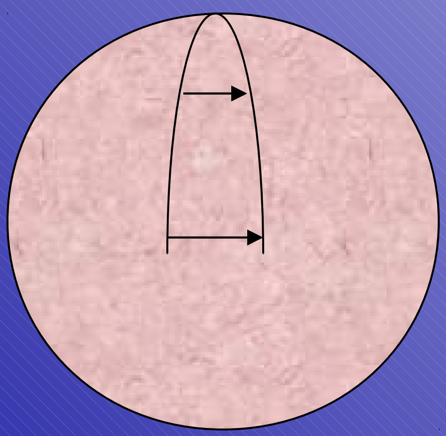
GW ASTRONOMY !!

Detection: Effect on test particles

Free test particles move along geodesics

Curvature and geodesics:

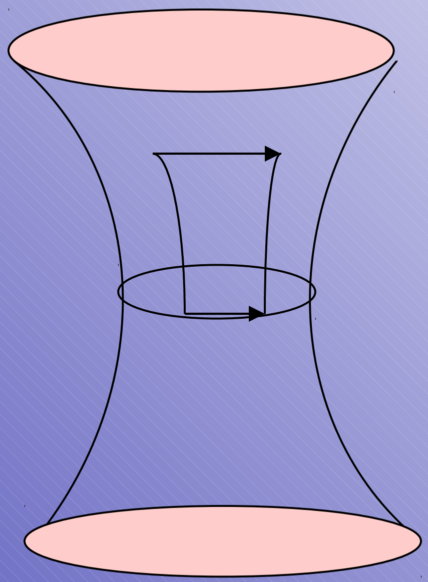
Sphere



$K > 0$

Geodesics move closer

Hyperboloid



$K < 0$

Geodesics move away

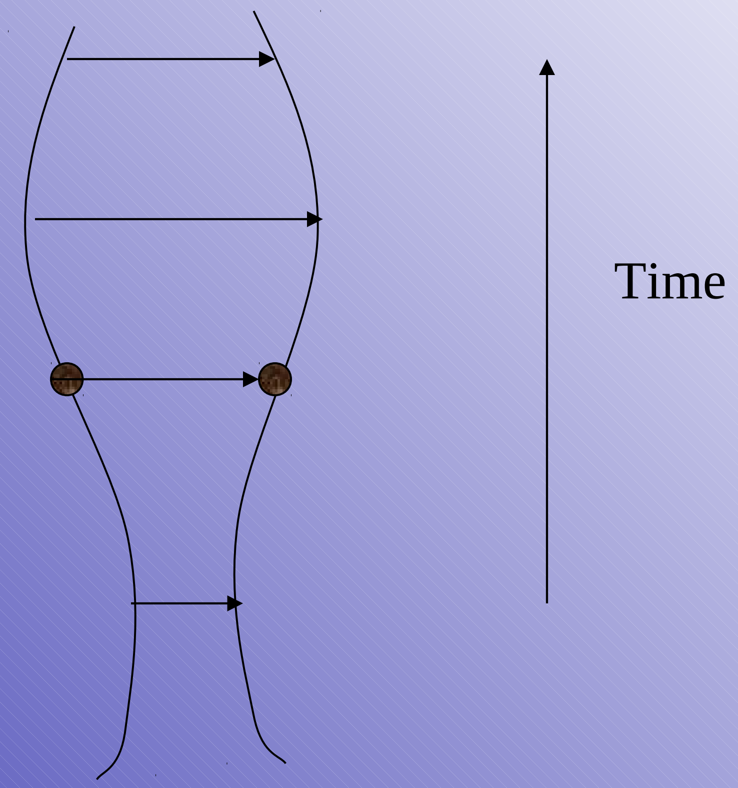
Gravitational wave and geodesics

- Gravitational wave

→ curvature oscillates

- The sign of the curvature keeps flipping

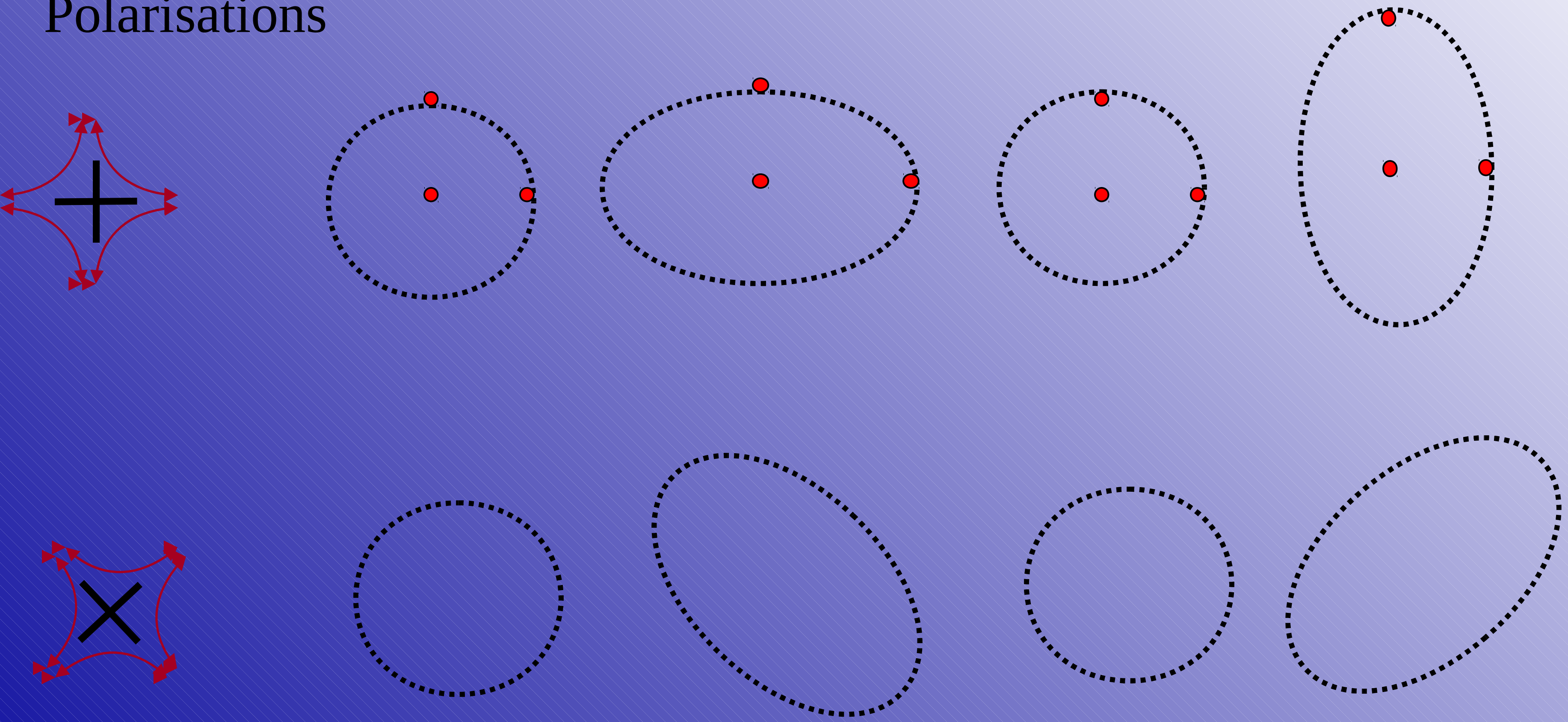
- The length of the connecting vector oscillates



$$R^{(0)}{}_{hijk}\ e^{-i\omega t}$$

Effect on a ring of test particles

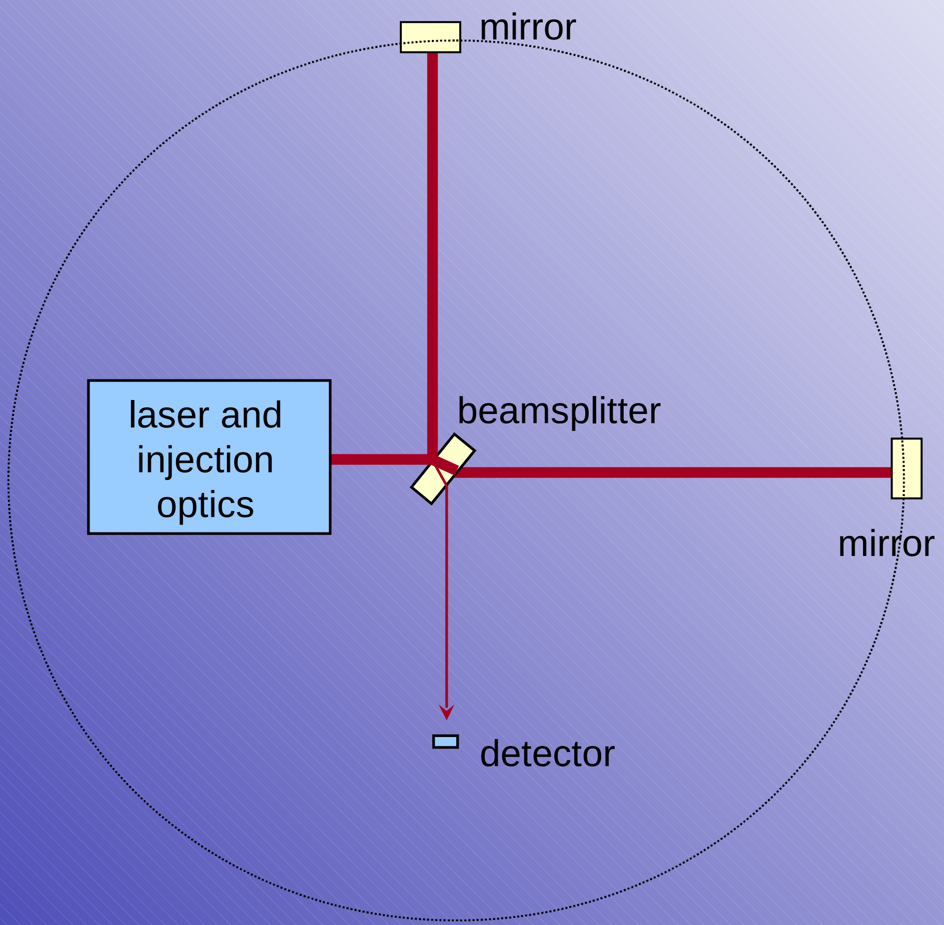
Polarisations



General wave: Linear combination of plus and cross

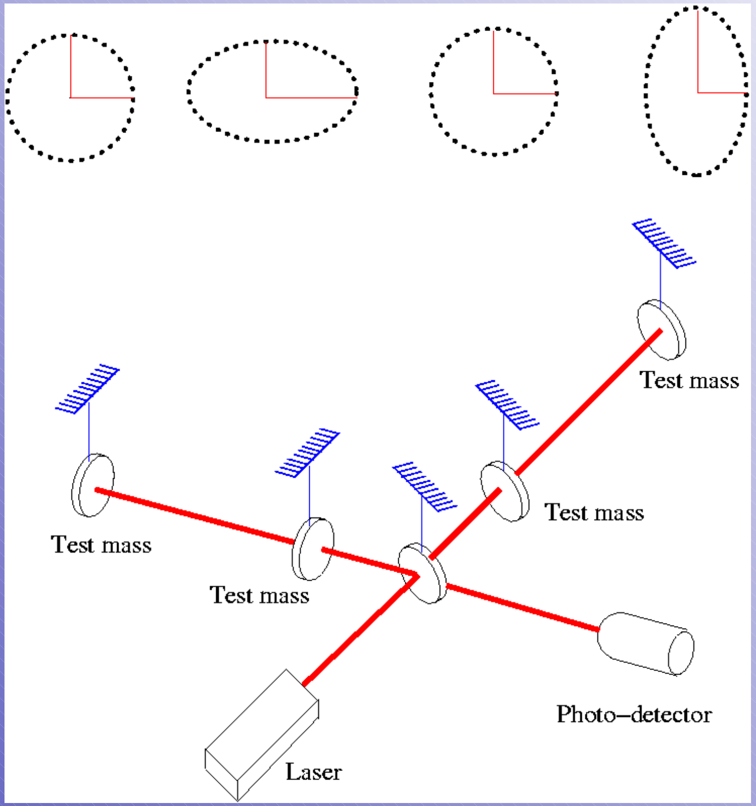
Detection principle on ground

Michelson interferometer measuring changes in relative lengths of arms formed between "free" test mass



PRINCIPLE OF DETECTION

- Quadrupolar force field
- Effect on a ring of test particles:



$$\delta L \sim h\, L$$

Quadrupole formula :

$$h \sim \left(\frac{E^{kin-ns}}{0.1 M\ c^2}\right)\left(\frac{R}{100\ Mpc}\right)^{-1} 10^{-21}$$

LIGO Louisiana 4 km armlength (US)

VIRGO 3 km armlength Pisa, Italy

CURRENT DETECTOR STATUS

- Several large scale laser interferometric detectors constructed: armlength of 300 m to 4 km

- LIGO, VIRGO, GEO, TAMA, AIGO, LCGT

- LIGO, VIRGO, TAMA, GEO : already taking data

Science runs

- Space based detector LISA – 5 million km

- Launch date 2020

International Network of GW Interferometers

1. Detection confidence 2. Source direction 3. Polarisation info

GEO: 0.6km

VIRGO: 3km

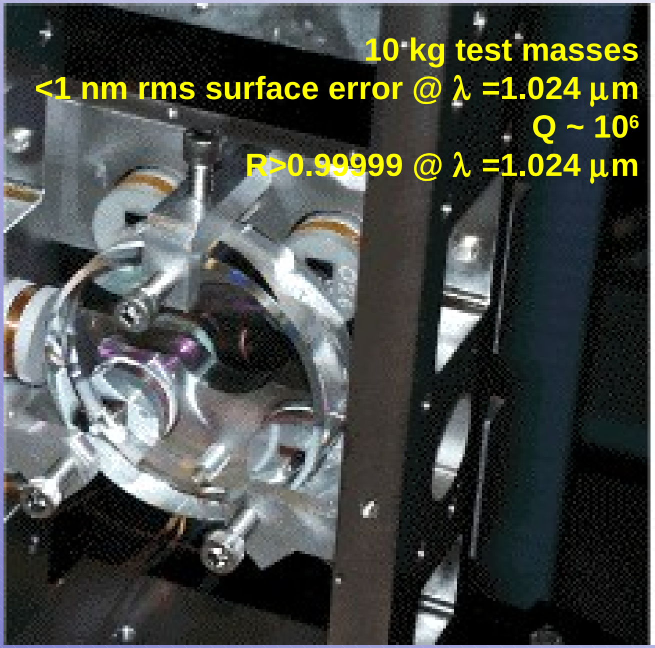
LIGO-LHO: 2km, 4km

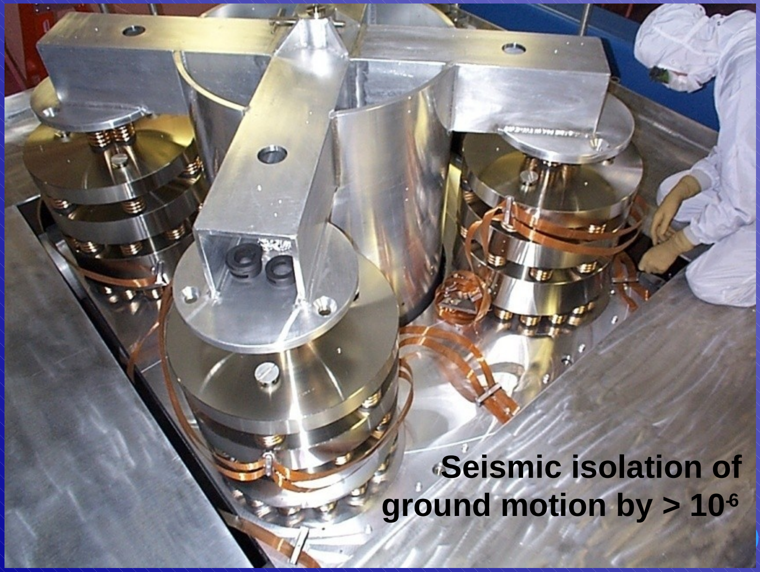
TAMA: 0.3km

LIGO-LLO: 4km

AIGO: (?)km

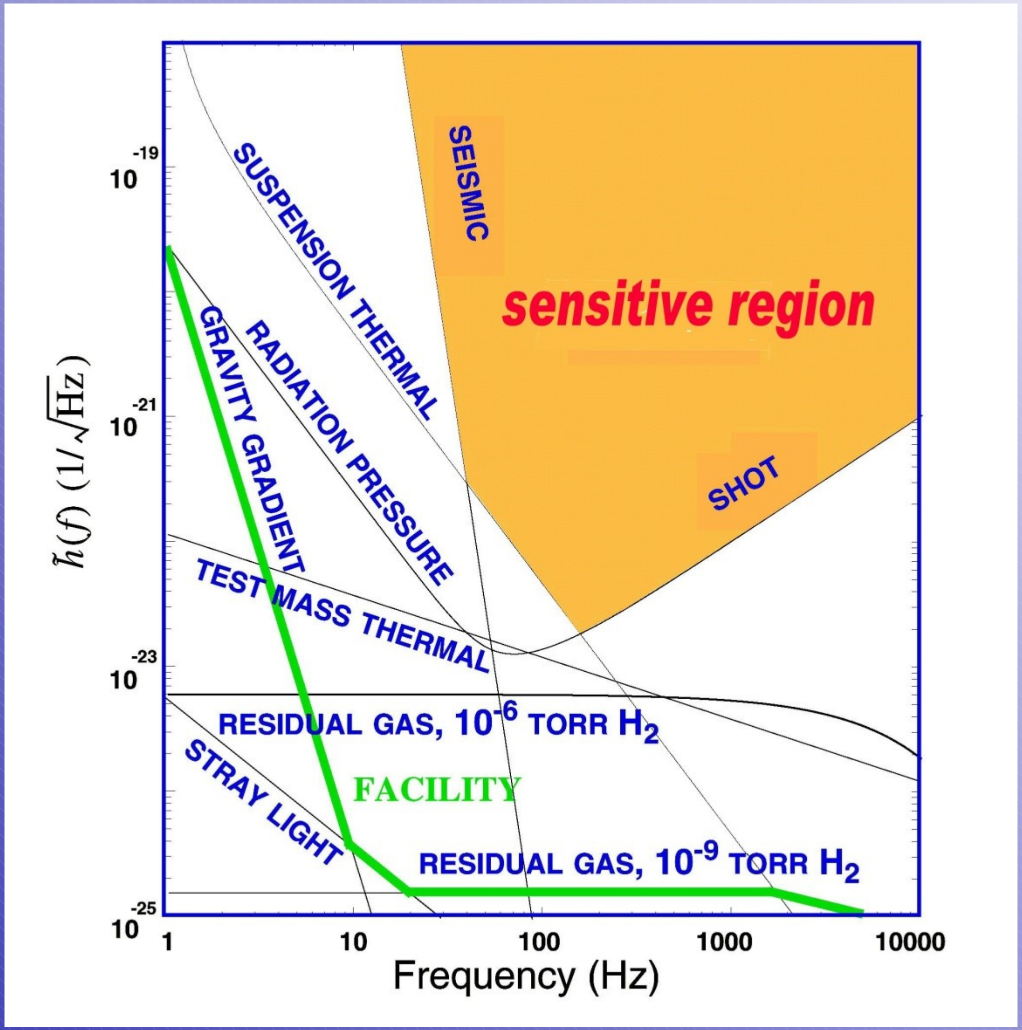
Technology pushed to the limits



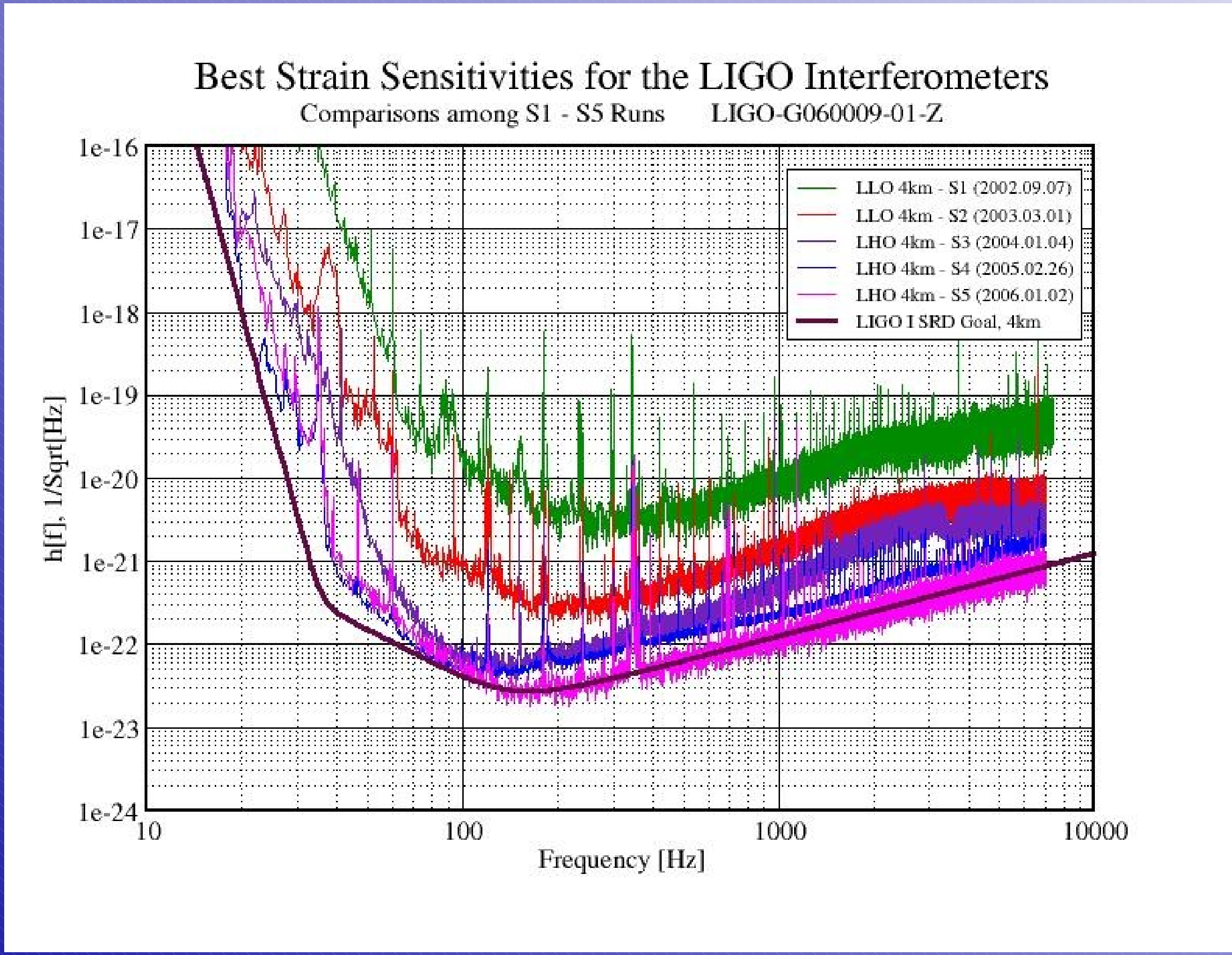


The noise floor

- Seismic noise at low frequencies
- Thermal noise at mid frequencies
- Shot noise at high frequencies – quantum nature of light



We did it !



*http://www.ligocaltech.edu/~lazz/distribution/LSC_Data

GRAVITATIONAL WAVE SOURCES

- Inspiraling binaries:

  Neutron stars (NS), Blackholes

  $h \sim 10^{-23}$ for 2 NS at 200 Mpc

- Rotating NS, Accreting NS – LMXBs

  Sco X-1

- Supernovae

- Stochastic background – Early Universe

  Parametric amplification

Source Strengths

Binary inspiral :

$$h \sim 2.5\times10^{-23} \left[\frac{M}{M_{sun}}\right]^{5/3} \left[\frac{r}{100\ Mpc}\right]^{-1} \left[\frac{f_a}{100\ Hz}\right]^{2/3}$$

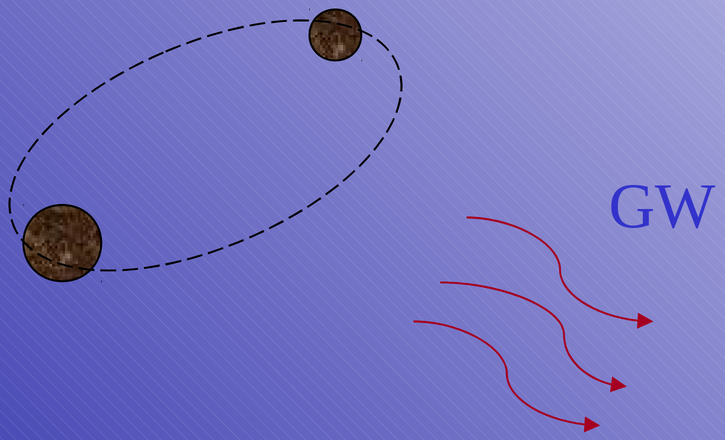
Periodic:

$$h \sim 1.9\times10^{-25} \left[\frac{I}{10^{45}\ gm.cm^2}\right] \left[\frac{f}{500\,Hz}\right]^{2} \left[\frac{r}{10\ kpc}\right]^{-1} \left[\frac{\varepsilon}{10^{-5}}\right]$$

Stochastic background:

$$\tilde{h}(f) \sim 10^{-26} \left[\frac{f}{10\ Hz}\right]^{-3/2} \left[\frac{\Omega_{GW}(f)}{10^{-12}}\right]^{1/2}$$

Inspiraling compact binaries



- Most promising source for interferometric detectors
- Waveform is well modeled by PN approximations

Matched filtering the signal

Waveform well modeled: The matched filter

$$c(\tau) = \int x(t)\, q(t+\tau)\, dt$$

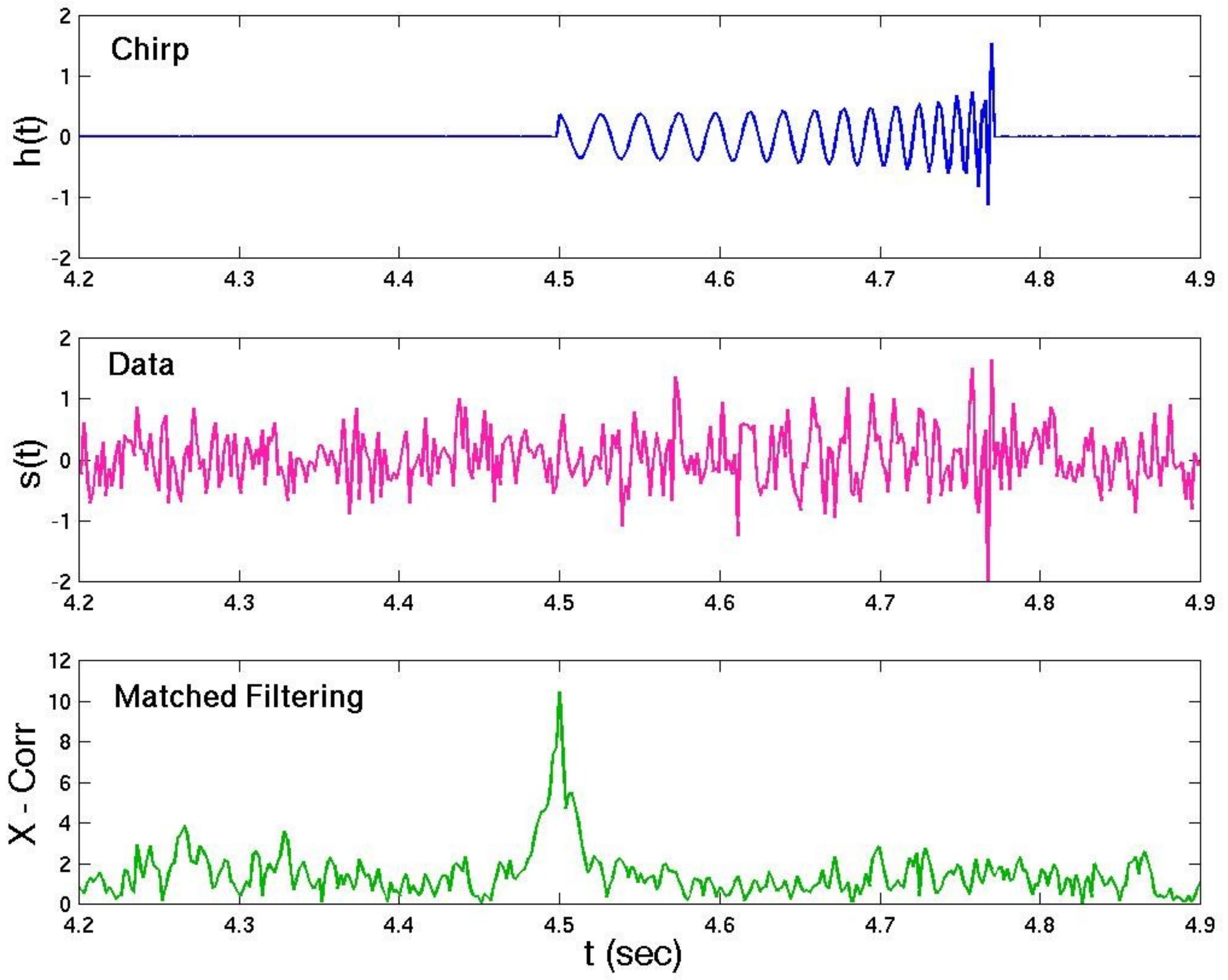
Stationary noise:

$$\tilde{q}(f) = \frac{\tilde{h}(f)}{S_h(f)}$$

Optimal filter in Gaussian noise:

Detection probability is maximised for a given false alarm rate

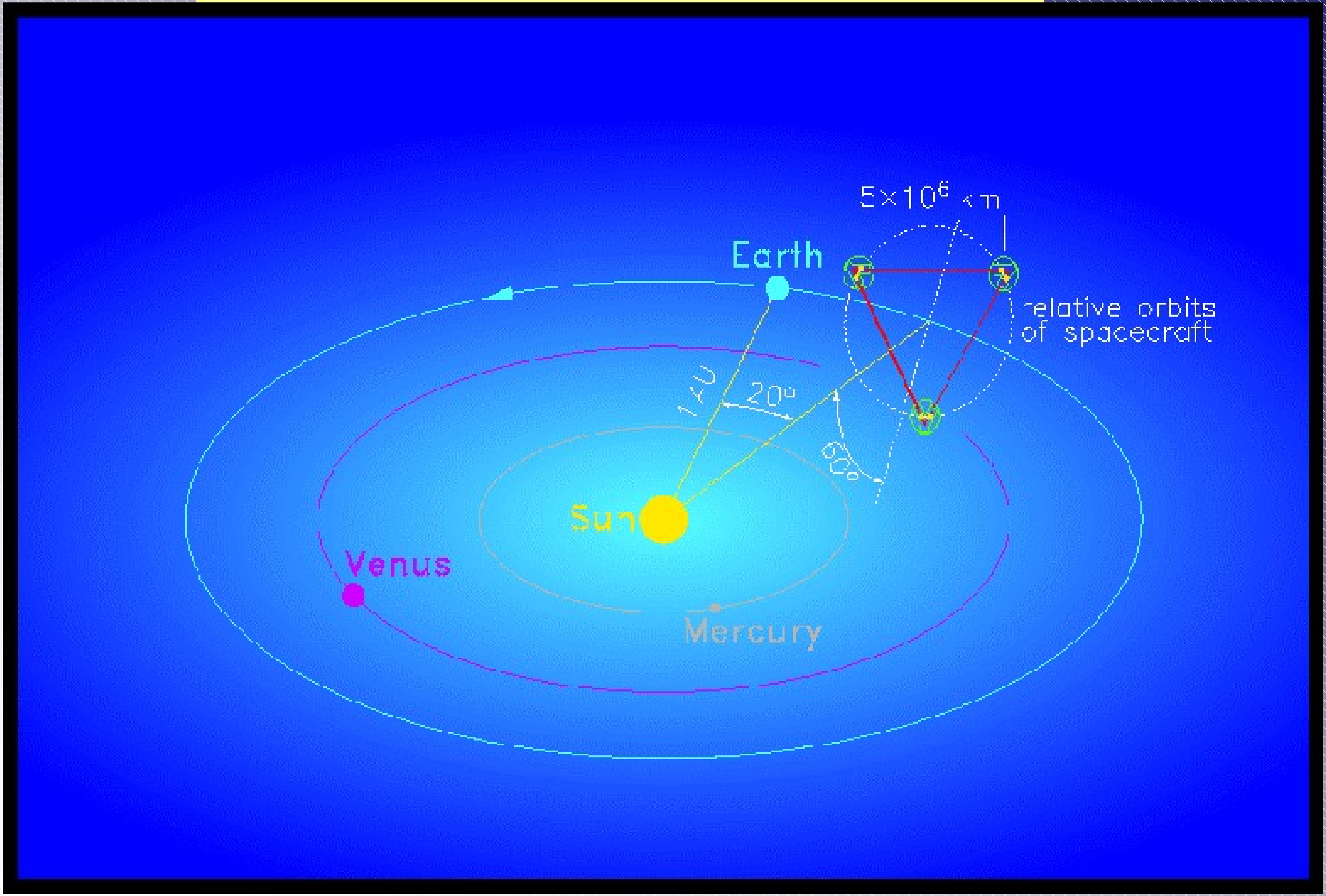
Matched filtering the inspiraling binary signal



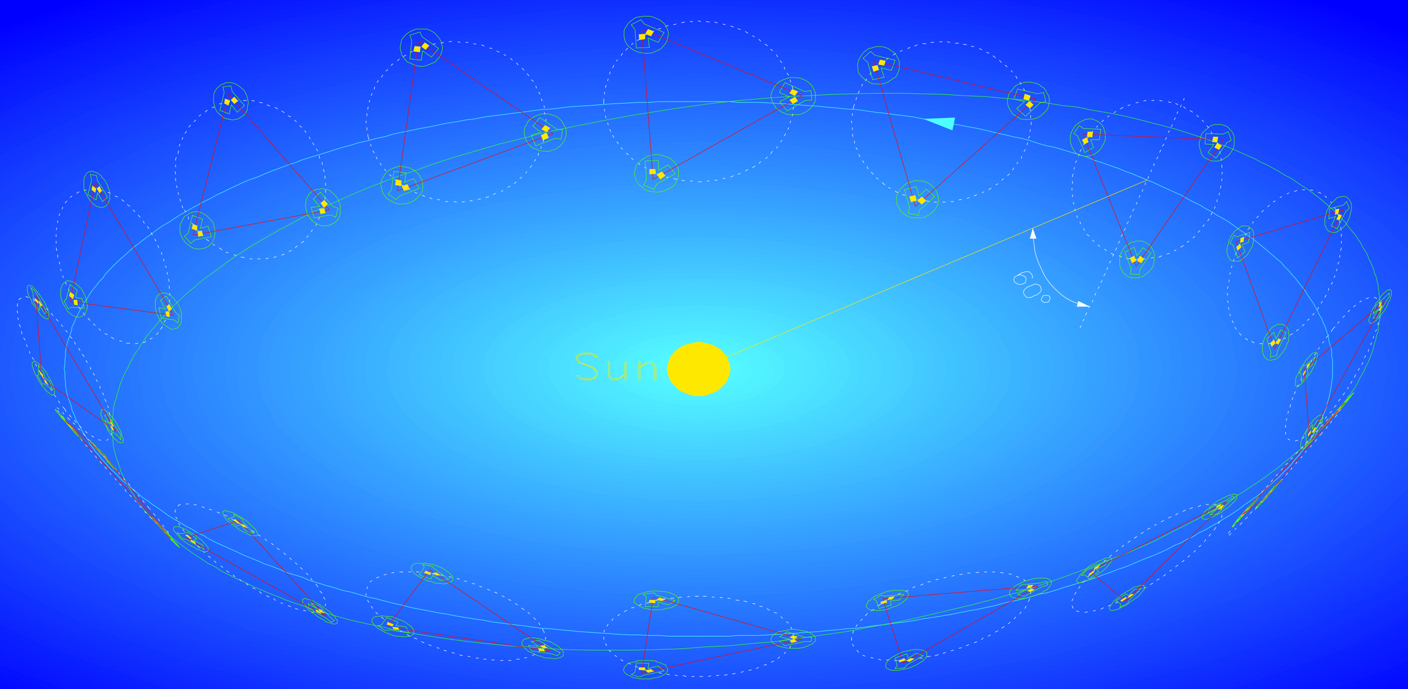
LISA

- **A Collaborative ESA / NASA Mission to observe low-frequency gravitational waves**
- **Cluster of 3 S/C in heliocentric orbit at 1 AU**
- **S/C contain lasers and free-flying test masses**
- **Equilateral triangle with 5 Million km arm-length**
- **Trailing the earth by 20°**
- **Equivalent to a Michelson interferometer**
  - Thermal & seismic motions of mirror masses and pendulums

THE LISA PROJECT

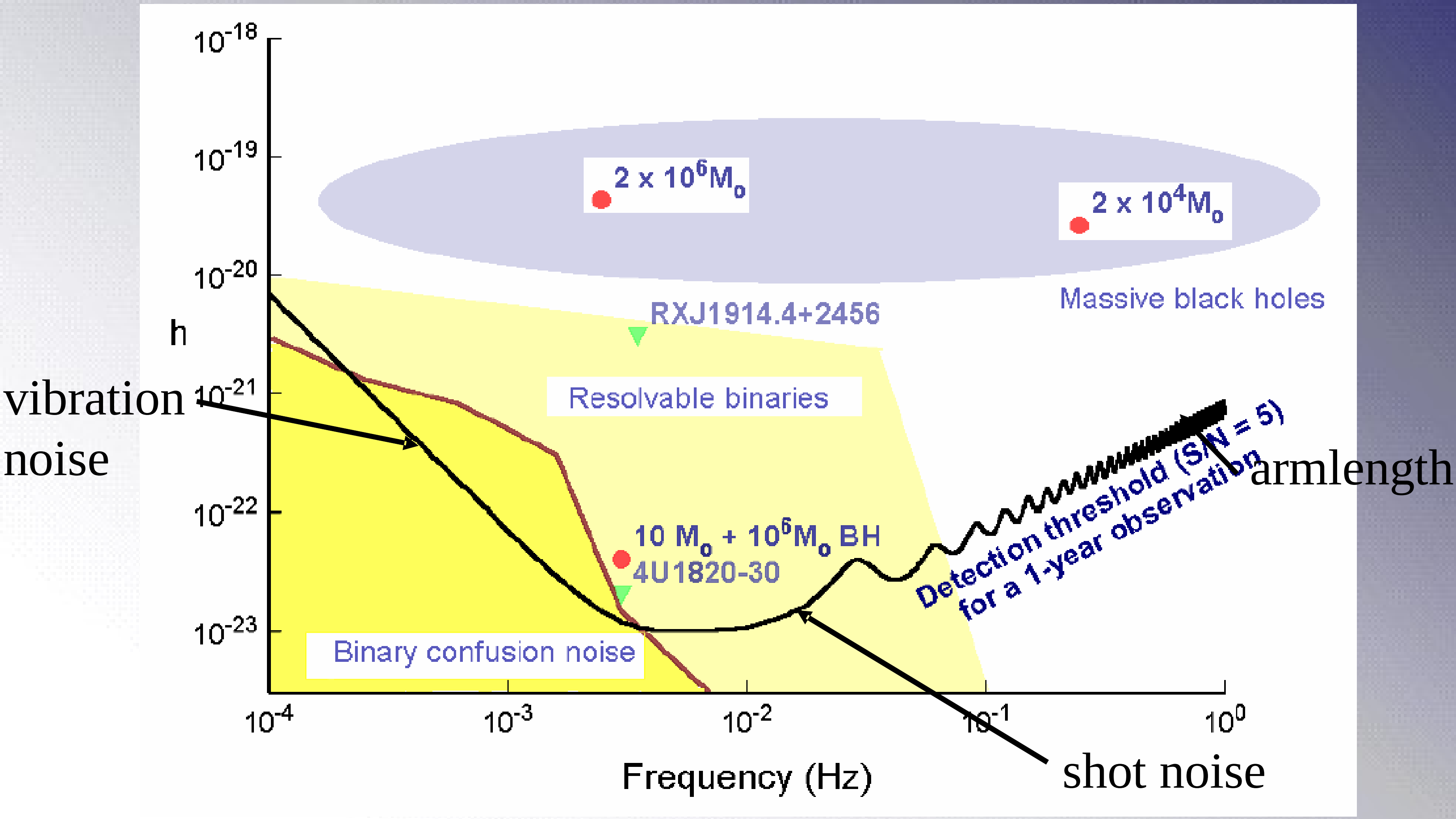


Orbit of LISA



Cluster rolls once per year around its centre

LISA noise curve



LISA SCIENCE

Fundamental Physics:

• Tests of strong field GR by mergers of comparable mass BHs:

- Area theorem – before/after measurements of M and J
- Cosmic Censorship – is a/M > 1 after merger?

- Stellar mass BH falling into a massive BH – few per year upto

1 Gpc (Sigurdson and Rees, Phinney) - EMRIs
High SNR - Test no-hair theorem ~ detailed waveforms with
10^5 cycles in the last year $10\ M_\odot$ falling into $10^6\ M_\odot$ BH

• Observe GW bursts from cosmic strings or other exotic sources

LISA SCIENCE contd

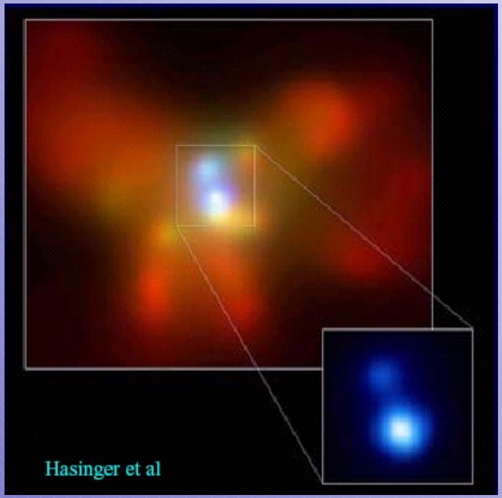
Astrophysics:

- Detect BH mergers in the range 10^5 - 10^7 $M_\odot$

- Study compact WD binaries
  - obtain mass spectrum …

- Detect hundreds of EMRIs
  - obtain spectrum of masses, spins

- Discover unexpected sources, dark matter components



FUTURE DIRECTIONS

- Future bright: 10 Hz to kHz

  Network of LIGO, VIRGO, GEO, LCGT, …

- Sensitivity upgrades: Amplitude, Band-width

  (For example: LIGO improved by factor ~ 100 just in amplitude sensitivity in 2 years)

  Initial detectors upto 2008

  Advanced detectors > 2008

- LISA will open the low frequency window

  10^{-4} Hz – 1 Hz

  Many detections! High SNR