A Rough Guide to Gravitational Radiation

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Outline

Aim is to provide an introduction to Gravitational Waves (GWs) without using General Relativity

- What are GWs ?
- Rough estimates for GW amplitude & luminosity associated with two promising sources
- How does the emission of GWs affect its source ?

GWs: I

General Relativity (GR) defines GWs as ripples in the curvature of space-time that propagate with the speed of light !

It is possible to **COMPUTE** most of the crucial effects of GWs using Newtonian Gravitational Theory, Classical Electrodynamics & some elements of Special Relativity

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- Newtonian gravity involves a scalar potential $\phi_{\rm N}(\mathbf{x}, t)$ such that $\nabla \phi_{\rm N} = 4\,\pi\, G\, \rho$, such that

$$\phi_{\rm N}(\mathbf{x}, t) = -G \int \frac{\rho(\mathbf{y}, t)}{r}\, d^3 y\,, \quad r \equiv |\mathbf{x} - \mathbf{y}| \tag{1}$$

  A change in $\phi_{\rm N}(\mathbf{x}, t)$ due to a change in $\rho(\mathbf{y}, t)$ propagate instantaneously

- Special Relativity demands that no information should be able to propagate faster than c: the speed of light

GWs: II

To make ϕ_{N} consistent with Special Relativity, we modify it

- $$\phi_{\mathrm{R}}(\mathbf{x}, t) = -G \int \frac{\rho(\mathbf{y}, t - \frac{r}{c})}{r} \, d^3y \tag{2}$$

  Dominant effects of GWs can be deducted from such a retarded gravitational potential

  This simple insertion will help us to define GWs in a non-rigorous way

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  This simple insertion will help us to define GWs in a non-rigorous way
- $\phi_{\rm R}$ satifies the scalar wave equation
  $\Box\phi_{\rm R} \equiv \left(\nabla^2 - \frac{\partial^2}{c^2\,\partial^2 t}\right)\,\phi_{\rm R} = 4\,\pi\,G\,\rho$
- Take the spatial gradient of $\phi_{\rm R}$

$$\boldsymbol{\nabla}\phi_{\rm R} = G \int \left(\frac{\rho}{r} - \frac{\partial\rho}{c\,\partial t}\right)\frac{\mathbf{x}-\mathbf{y}}{r^2}\, d^3y \tag{3}$$

GWs: III

- If $|\mathbf{x}| \gg |\mathbf{y}_{\rm mx}|$, we have $r \sim |\mathbf{x}|$ & can negelct $1/r$ term in the previous Eq.

$$\mathbf{n} \cdot \boldsymbol{\nabla} \phi_{\rm R} \sim \frac{\partial \phi_{\rm R}}{c\,\partial t}\,, \;\; \mathbf{n} = \mathbf{x}/|\mathbf{x}| \tag{4}$$

$$\phi_{\rm R}/\lambda \sim 1/c \times \phi_R/T$$

- If we are far away from a GW source, the typical length scale over which $\phi_{\rm R}$ varies is **c** $\times$ the typical time scale over which $\phi_{\rm R}$ changes

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- This is true for a wave traveling at speed c
  This is our Gravitational Wave

- Recall that $\phi_R \sim v^2$. Therefore, the amplitude of GW should be $\sim$

$$h \sim \frac{\left(time - dependent\ part\ of\ \phi_R \right)}{c^2} \tag{5}$$

An estimate for the dominate contribution to h: I

Consider a region $|\mathbf{x}| \gg |\mathbf{y}| b$ & we have far-zone expansion

- $$1/r \equiv |\mathbf{x} - \mathbf{y}|^{-1} \sim \frac{1}{|\mathbf{x}|} + \mathbf{y} \cdot \mathbf{n}\, |\mathbf{x}|^{-2} \tag{6}$$

  This leads to (if we neglect $\mathcal{O}(|\mathbf{x}|^{-2})$ terms)

  $$\phi_{\mathrm{R}} = -\frac{G}{|\mathbf{x}|} \int \rho(\mathbf{y}, t_r)\, d^3y\,, \quad t_r = t - \frac{r}{c} \tag{7}$$

- Let $t_0 = t - \frac{|\mathbf{x}|}{c}$ & this leads to $t_r \sim t_0 - \mathbf{y} \cdot \mathbf{n}/c$
- Expand $\rho(t_r)$ about t_0

$$\phi_R = \frac{-G}{|\mathbf{x}|} \left\{ \int \left[\rho(t_0) - \frac{\dot{\rho}}{c}\, \mathbf{n} \cdot \mathbf{y} + \frac{\ddot{\rho}}{2\, c^2}\, (\mathbf{n} \cdot \mathbf{y})^2 + ... \right] d^3y \right\} \tag{8}$$

An estimate for the dominate contribution to h: II

- First term

$$\int \rho(t_0) d^3y \equiv M \tag{9}$$

- Let $\mathbf{v} = \frac{d\mathbf{y}}{dt}$; the second term involves

$$n_i \int \dot{\rho} y_i \, d^3y = n_i \int \rho \, v_i \, d^3y = \mathbf{n} \cdot \mathbf{P} \tag{10}$$

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- Third term contains

$$\int \ddot{\rho} \, y_i \, y_j \, d^3 \, y = \ddot{I}_{ij} \tag{11}$$

  where $I_{ij}(t) \equiv \int \rho(t) \, y_i \, y_i \, d^3 y$ is the quadrupole moment of gravitating source & $\ddot{I}_{ij} = \int \rho \, v_i \, v_j \, d^3 y$
- Retarded potential becomes

$$\phi_R \sim -\frac{G\,M}{|\mathbf{x}|} + \frac{G\,\mathbf{n} \cdot \mathbf{P}}{c\,|\mathbf{x}|} - \frac{G}{2\,c^2} \frac{\ddot{I}_{ij}\,n_i\,n_j}{|\mathbf{x}|}\,, \tag{12}$$

An estimate for the dominate contribution to h: III

- This leads to

$$h \sim \frac{G}{2\,c^4}\,\frac{\ddot{I}_{ij}\,n_i\,n_j}{|\mathbf{x}|} \tag{13}$$

- Note that h depends only on the components of I_{ij} along $\mathbf{n}$, the direction of propagation of the wave & this is due to the fact that we are dealing with **scalar waves**

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- In GR, GWs are *ripples in the curvature of space-time* & space-time & its disturbances are described by tensors
- $I_{ij} \rightarrow$ transverse components of trace-free tensor $\mathcal{I}_{ij} = I_{ij} - \frac{\delta_{ij}}{3}\,I_{kk}$
- In GR, we have
$$h_{ij}^{TT} = \frac{2\,G}{c^4\,r}\,\ddot{\mathcal{I}}_{ij}(t - r/c) \tag{14}$$

  This implies that spherically symmetric motion **WILL NOT** produce GWs. Any spherically symmetric tensor $\propto \delta_{ij}$ & hence $\mathcal{I}_{ij}$ vanishes

An estimate for GW luminosity :I

- In classical ED, the dominant order multipole radiation fron a charge distribution is the dipole radiation. The vector potential A_i in the wave-zone

$$A_j = \frac{1}{c\, r}\, \dot{d}_j(t_r) \tag{15}$$

The $1/r$ EM fields $\mathbf{E}$ & $\mathbf{B}$ depend only on the components of $\dot{\mathbf{d}}$ transverse to $\mathbf{n}$; $d_j^{\mathrm{T}} \equiv P_{jk}\, d_k\,,\ \ P_{jk} = \delta_{jk} - n_j\, n_k$

- The Larmor formula provides the expression for EM luminosity

$$\mathcal{L}_{\mathrm{EM}} = \frac{2}{3\, c^3}\, \ddot{d}_j\, \ddot{d}_j\,,\ \ d_j = e\, y_i \tag{16}$$

- For gravitating systems, linear & angular momenta provide electric & magnetic *type* dipole moments & they are conserved $\boldsymbol{\mu} = \frac{1}{c} \sum_a \mathbf{y}^a \times \mathbf{d}^a = \frac{1}{c} \sum_a \mathbf{y}^a \times m_a \mathbf{v}^a = \frac{1}{c} \sum_a \mathbf{L}^a$

An estimate for GW luminosity :II

$\mathcal{L}_{\rm GW} \propto \overset{(3)}{I_{ij}}\,\overset{(3)}{I_{ij}}$ & dimensional consideration require us to have G/c^5

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$$\mathcal{L}_{GW} = \frac{G}{5\,c^5}\,\mathcal{I}_{ij}^{(3)}\,\mathcal{I}_{ij}^{(3)} \tag{17}$$

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- ${}^{(3)}I_{ij} \sim M\,R^2/T^3 \sim M\,V^3/R$

$$\mathcal{L}_{GW} \sim \frac{G}{c^5}\,(M/R)^2\;V^6 \sim L_0\,(r_{Sch}/R)^2\,(V/c)^6\;, \tag{18}$$

where $L_0 = \frac{c^5}{G} \sim 3.6 \times 10^{52}\mathrm{J}/s$ & $r_{\rm Sch} = G\,M/c^2$

- $\mathcal{L}_{GW}$ is maximal if $R \sim r_{\rm Sch}$ & $V \sim c$

Compact objects, having time-dependent quadrupole moment, moving with velocities $\sim c$ are copious sources of GWs

How to detect GWs ?:I

- Recall that $h \sim \frac{G}{2\,c^4}\,\frac{\ddot{I}_{ij}\,n_i\,n_j}{|\mathbf{x}|}$ & $\ddot{I}_{ij} \sim \int \rho\, v_i\, v_j d^3y \sim M\,\phi_{\mathrm{int}}$
  ϕ_{int} provides typical value for Newtonian potential inside the source
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$$h \sim \frac{G\,M}{c^2\,r'}\,\frac{\phi_{\mathrm{int}}}{c^2} \sim \frac{\phi_{\mathrm{N}}}{c^2}\,\frac{\phi_{\mathrm{int}}}{c^2} \sim \frac{r_{\mathrm{Sch}}}{r'}\,\frac{v^2}{c^2} \tag{19}$$
- $h \ll \frac{\phi_{\mathrm{N}}}{c^2}$ and *it is not possible to detect even a nearby star by measuring its* ϕ_N!
- Recall that the acceleration **a** due to ϕ_{N} is $\sim \phi_{\mathrm{N}}/|\mathbf{x}|$
  However, **a** due to passing GW of amplitude h and wavelength λ is $\sim c^2\,h/\lambda$

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  However, **a** due to passing GW of amplitude h and wavelength λ is $\sim c^2\, h/\lambda$
- Entire Earth feels the above **a** & not possible to measure (Einstein's equivalence principle)

  However, it is possible to measure difference in the above **a** across an experiment **(Tidal force** $\sim M/L^3$**)**

How to detect GWs ?:II

- If the experiment has a size l, the tidal **a** due to a passing GW $\mathbf{a}_{\rm Tidal-GW} \sim l\, c^2\, h/\lambda^2 \sim h\, l\, \omega^2$, where ω being the angular frequency of the wave. **The above qty is** $\gg \phi_{\mathbf{N}}\, l/|\mathbf{x}|^2$

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- Consider two particles in an empty space in the presence of a passing GW & let $h = h_0\, e^{i\,\omega\,t}$. The Eq for the change in their separation δl

$$\ddot{\delta} l = \omega^2\, l\, h_0\, e^{i\,\omega\,t} \tag{20}$$

- We have $\delta l = \delta l_0\, e^{i\,\omega\,t}$ such that $|\delta l_0/l| = h_0$

  Therefore, h is the relative strain induced in a system of **free** particles by the passing GW

- In GR, distances along the direction of propagation are NOT affected due to transverse nature of tensorial GWs

GWs from non-spherical collapse as in supernovae

- GW energy radiated $\Delta E \sim \mathcal{L}_{\rm GW}\, T$

  $\mathcal{L}_{\rm GW} \sim \frac{c^5}{G}\, (r_{Sch}/R)^2\, (V/c)^6 \sim \frac{c^5}{G}\, (r_{Sch}/R)^5$

  $T \sim \left(\frac{R^3}{G\,M}\right)^{1/2} \sim \frac{1}{c}\, \left(\frac{R^3}{r_{Sch}}\right)^{1/2}$

- $\Delta E \sim M\, c^2\, (r_{Sch}/R)^{7/2} = \nu\, M\, c^2$

- GW amplitude becomes $h \sim (r_{Sch}/r')\, \left(V^2/c^2\right) \sim \nu^{2/7}\, \frac{r_{Sch}}{r'}$

$$h \sim 10^{-18}\, \left(\frac{\nu}{0.1}\right)^{2/7}\, \left(\frac{M}{M_{\odot}}\right)\, \left(\frac{r'}{10\,\mathrm{K}pc}\right)^{-1} \tag{21}$$

$h \sim 10^{-21}$ for a supernova ~ 20 Mpc & this is really an upperbound
GW frquencies are $\sim 100 \rightarrow 10^3$ Hz

GWs from compact binaries :I

For binaries, $h \sim \frac{\phi_{\rm N}}{c^2} \frac{\phi_{int}}{c^2}$ is fairly realistic estimate

- PSR 1913 + 16 The system contains a Pulsar being orbited by an unseen Neutron Star & having $M \sim 2.8\, M_{\odot}$, $a/c \sim 2$s ($P_{\rm orb} \sim 7\,{\rm hr}$) & $r' \sim 5$Kpc.

  $h \sim 10^{-23}$,$f_{GW} \sim 100\,\mu\,{\rm Hz}$ & $L_{GW} \sim 10^{24} J/s$

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- Compact binaries with $R \sim 100$ km are of great interest to LIGO/VIRGO
- Define τ_{GW} as the time it takes a binary to radiate half of its potential energy

$$\tau_{GW} = \frac{G\,M^2}{2\,R\,\mathcal{L}_{GW}}$$

GWs from compact binaries :II

Rough estimates for the amplitude, frequency & duration of inspiralling compact binaries are

$$h \approx 10^{-21} \left(\frac{15\,\mathrm{M}pc}{r} \right) \left(\frac{M}{2.8\,M_\odot} \right)^2 \left(\frac{90\,\mathrm{k}m}{R} \right), \tag{22}$$

$$f_{\rm GW} = \left(\frac{M}{2.8\,M_\odot} \right)^{1/2} \left(\frac{90\,\mathrm{k}m}{R} \right)^{3/2} 100\,\mathrm{H}z\,, \tag{23}$$

$$\tau_{GW} = \left(\frac{2.8\,M_\odot}{M} \right)^3 \left(\frac{R}{90\,\mathrm{k}m} \right)^4 0.5\,\mathrm{s}\,, \tag{24}$$

The radiating system become more compact, its amplitude & frequency increase

From measuring $h, f_{\rm GW}$ & τ_{GW}, one can estimate r, M, R & other quantities like i: the orbital inclination

On Radiation Reaction effects: I

- Consider a classical electron (e^-) orbiting a classical proton & emiting EM radiation. The system loses energy at a rate given by the Larmor formula & the orbits shrinks !
- This description is **incomplete !** If the e^- only feels the Coulomb field of the proton, its motion must remain **circular** & its inspiraling motion can not take place
- Therefore, we are forced to conclude that the e^- is subjected to *its own electric field !*

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- Therefore, we are forced to conclude that the e^- is subjected to *its own electric field !*
- The e^-'s own field should diverge at its position. Therefore, how can it produce a *finite Radiation Reaction force that drives the inspiral* The finite part of the electron's self field provides the force that drives the inspiral of e^-.
- It is fairly complicated to compute $\phi_{\rm RR}$ in GR such that $\mathbf{F}^{\rm Re} = -M\, \mathbf{\Delta} \phi^{\rm Re}$

On Radiation Reaction effects: II

- Recall our 'retarded gravitational potential'

$$\phi_{\mathrm{R}}(\mathbf{x}, t) = G \int \frac{\rho(\mathbf{y}, t - \frac{r}{c})}{r} \, d^3y \tag{25}$$

Taylor expand $\rho(t - \frac{r}{c})$ around $\rho(t)$ (near-zone expansion)

$$\phi_{\mathrm{R}} = -G \int r^{-1} \sum_{n=0}^{\infty} \left(-\frac{r}{c}\right)^n \frac{1}{n!} \frac{d^n}{dt^n} \rho(\mathbf{y}, t) \, d^3y \tag{26}$$

- We will need to go to the 6^{th} term to get an estimate for ϕ^{Re}
Terms with $n = 0, 2, 4$ provides non-vanishing contributions to ϕ_{R} & they provide contributions to the conservative dynamics: the so-called Newtonian, 1PN & 2PN corrections to the dynamics
Terms associated with $n = 1, 3$ should vanish
- $n = 5$ term is $\frac{G}{120\, c^5} \int r^4 \, \rho^{(5)} \, d^3y$

On Radiation Reaction effects: III

- $$\phi^{\mathrm{Re}}(\mathbf{x},t) = \frac{G}{30\,c^5}\left\{ \overset{(5)}{I}_{ij}\,x_i\,x_j + \frac{1}{2}\,|\mathbf{x}|^2\,\overset{(5)}{I}_{kk} - x_i\,T_i \right\} \tag{27}$$

  where $T_i = \int \rho\, y_i\, |\mathbf{y}|^2\, d^3y$
- These terms arise from $r^4 = \left(|\mathbf{x}|^2 - 2\,\mathbf{x}\cdot\mathbf{y} + |\mathbf{y}|^2\right)^2$
  Terms that matter are $4\,(\mathbf{x}\cdot\mathbf{y})^2, 2\,|\mathbf{x}|^2\,|\mathbf{y}|^2, -4\,(\mathbf{x}\cdot\mathbf{y})\,|\mathbf{y}|^2$ as the rest may be ignored !
- ϕ^{Re} is the only term that can do any work on the system

  $$\frac{dE}{dt} = -\int \rho\, v_i\, \nabla_i\, \phi_{\mathrm{R}}\, d^3x = -\int \dot{\rho}\, \phi_{\mathrm{R}}\, d^3x \tag{28}$$

  Contributions $\phi^{\mathrm{N}}, \phi^{\mathrm{1PN}}, \phi^{\mathrm{2PN}}$ provide only total time derivatives to $\frac{dE}{dt}$ & hence $\rightarrow 0$ on orbital averaging !

On Radiation Reaction effects: IV

- $$-\left\langle \int \dot{\rho}\, \phi^{\mathrm{Re}}\, d^3x \right\rangle = -\frac{G}{30\, c^5} \left\langle \dot{I}_{ij}\, {}^{(5)}I_{ij} + \frac{1}{2}\, {}^{(3)}I_{kk}\, {}^{(3)}I_{kk} \right\rangle \tag{29}$$

- In GR, under post-Newtonian approximation, final results are more compact

$$\left\langle \frac{dE}{dt} \right\rangle = -\frac{G}{5\, c^5} \left\langle {}^{(3)}\mathcal{I}_{ij}\, {}^{(3)}\mathcal{I}_{ij} \right\rangle \tag{30}$$

$$\phi^{\mathrm{Re}} = \frac{G}{5\, c^5}\, {}^{(5)}\mathcal{I}_{ij}\, x_i\, x_j \tag{31}$$

References

- *Gravitational waves on the back of an envelope*, **B. F. Schutz** American Journal of Physics, Volume 52, (5) pp. 412-419 (1984).

- *Black Holes, White Dwarfs and Neutron Stars: The Physics of Compact Objects* , **S. L. Shapiro & S. A. Teukolsky**